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Localization of the Kobayashi Distance

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1. INTRODUCTION

Given a domain $\Omega \subseteq \mathbb{C}^n$, we denote by $F_K^\Omega(z, \xi)$ the infinitesimal form of the Kobayashi metric for Ω at z in the direction of the vector ξ. Then it follows from results in [1, 3, 5] that if ξ is fixed and z is allowed to approach a strongly pseudoconvex point P in the boundary of Ω, we obtain the estimate

$$F_K^\Omega(z, \xi) \approx c \frac{|\xi_{N_P}|}{\delta_\Omega(z)} + c \frac{|\xi_{T_P}|}{\sqrt{\delta_\Omega(z)}} \quad \text{for all} \quad z \in U \cap \Omega, \tag{*}$$

where U is a neighborhood of P where the eigenvalues of the Levi form at P are bounded from zero and for any $\xi \in \mathbb{C}^n$, ξ_{N_P} is the complex normal component of ξ at P and ξ_{T_P} is the complex tangential component of ξ at P and $\delta_\Omega(z)$ is the distance from z to the boundary.

By means of the estimate (*) is possible to solve the following problem:

THEOREM 1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with at least C^{n+1} boundary and $P \in \partial\Omega$ be strongly pseudoconvex point and let U and V be small neighborhoods of P in $\mathbb{C}^n$ with $V \subset\subset U$ such that the eigenvalues of the Levi form on $\partial\Omega$ are bounded from 0 on $U \cap \partial\Omega$. Then there is a constant $C = C(U, V, \Omega)$ such that*

$$1 \leqslant \frac{K_{\Omega \cap U}(z_0, w_0)}{K_\Omega(z_0, w_0)} \leqslant C,$$

for all z_0 and w_0 in $V \cap \Omega$.

We point out that Royden got the following estimate which appears in [3] as Lemma 22 without proof.

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LEMMA. *Let V and U be subdomains of a hyperbolic manifold M. For $x \in U$, we define*

$$d^*(x) = d_V^*(x, V \sim U) = \inf_{y \in V \sim U} d^*(x, y),$$

where $d^*(x, y) = \inf\{\rho(a, b) : \exists f\colon \Delta \to V$ holomorphic, $f(a) = x$ and $f(b) = y\}$ and ρ is the Poincaré metric in the unit disc Δ in $\mathbb{C}$.

Then

$$F_K^V(x, \xi) \leqslant F_K^{U \cap V}(x, \xi) \leqslant \coth(d^*(x))\, F_K^V(x, \xi).$$

2. NOTATIONS AND DEFINITIONS

DEFINITION 2.1. If $e_1 = (1 + 0i, 0, ..., 0)$ then the infinitesimal form of the *Kobayashi metric* for Ω at z in the direction of ξ is

$$F_K^\Omega(z, \xi) = \inf\left\{\frac{|\xi|}{|(f_*(0))(e_1)|} : f\colon B \to \Omega \text{ is holomorphic, } f(0) = z, \right.$$
$$\left. \text{and } (f_*(0))(e_1) \text{ is a constant multiple of } \xi\right\}.$$

DEFINITION 2.2. The *Kobayashi distance* between the points z, $w \in \Omega$ can be defined as

$$K_\Omega(z, w) = \inf_\gamma \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))\, dt,$$

where the infimum is taken over piecewise differentiable curves $\gamma\colon [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

For details about the metric and pseudoconvex domains see [2].

The following theorem has been proven in [1] and is a basic tool for our future calculation.

THEOREM 2.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with C^{n+1} boundary. Suppose $P \in \partial\Omega$ is a strongly pseudoconvex point and W is a neighborhood of P on which the eigenvalues of the Levi form are bounded from zero by some number $\varepsilon > 0$. Let us assume without loss of generality that z_1 is the normal complex direction at P. Let ρ be a defining function for Ω such that $|\nabla_z \rho(w)| = 1$ for all $w \in \partial\Omega$. Let Q be a unitary operator which diagonalizes the Levi form at P, and let $\lambda_1, ..., \lambda_n$ be the eigenvalues of the*

Levi form at P where the corresponding eigenvectors have, respectively, the directions $z_1, \ldots, z_n$. Let $z \in \Omega$ and S be the projection of z into $\partial\Omega$. Given $\xi \in \mathbb{C}^n$, let ξ_{N_S} be the complex normal component of ξ at S and ξ_{T_S} be the complex tangential component of ξ at S. Define

$$\eta(z) = \sqrt{2}\delta_\Omega(z)\xi_{N_S} + \sqrt{\delta_\Omega(z)}\, H(Q\xi_{T_S}),$$

where H is the diagonal matrix with entries $\lambda_i^{-1/2}$.
Then

$$\lim_{\Omega \ni z \to P} F_{\mathrm{K}}^{\Omega}(z, \eta(z)) = \frac{1}{\sqrt{2}} |\xi|.$$

As a consequence of the theorem we obtain the estimate

$$F_{\mathrm{K}}^{\Omega}(z, \xi) \approx c \frac{|\xi_{N_P}|}{\delta_\Omega(z)} + c \frac{|\xi_{T_P}|}{\sqrt{\delta_\Omega(z)}} \qquad \text{for all} \quad z \in U \cap \Omega,$$

where P is a strongly pseudoconvex point in the boundary of a pseudoconvex domain Ω, U is a neighborhood of P where the eigenvalues of the Levi form at P are bounded from zero, and for any $\xi \in \mathbb{C}^n$, ξ_{N_P} is the complex normal component of ξ at P and ξ_{T_P} is the complex tangential component of ξ at P.

3. Proof of Theorem 1

It is obvious that $K_\Omega(z_0, w_0) \leqslant K_{\Omega \cap U}(z_0, w_0)$ for all z_0 and w_0 in $V \cap \Omega$. Let us prove that $K_\Omega(z_0, w_0) \geqslant CK_{\Omega \cap U}(z_0, w_0)$.

Assume $\delta_\Omega(z_0) \leqslant \delta_\Omega(w_0)$ and let $\varepsilon = \max\{|z_0 - w_0|^2, \delta_\Omega(z_0), |\pi_P(w_0 - z_0)|\}$, where $\pi_P\colon \mathbb{C}^n \to \mathcal{N}_P$ is the orthogonal projection into $\mathcal{N}_P$, the complex normal space to $\partial\Omega$ at P. Let us define the pseudoballs

$$Q_1(z_0, \varepsilon) = \{z \in \Omega\colon |\pi_P(z - z_0)| < \varepsilon;\ |z - z_0| < \sqrt{\varepsilon}\}$$

and

$$Q_2(z_0, 10/9\varepsilon) = \{z \in \Omega\colon |\pi_P(z - z_0)| < 10/9\varepsilon;\ |z - z_0| < \sqrt{10/9\varepsilon}\}.$$

Let γ be a curve which connects z_0 with w_0 and parametrized with respect to arc length, with $L_{\mathrm{Euc}}(\gamma) = t_0$.

We have two possibilities:

(i) $\delta_\Omega(w_0) \geqslant \delta_\Omega(\gamma(t)) \geqslant \delta_\Omega(z_0)$, for all t

(ii) $\delta_\Omega(\gamma(t)) > \delta_\Omega(w_0)$ or $\delta_\Omega(\gamma(t)) < \delta_\Omega(z_0)$, for some t.

Case (i). Let $t_1 \in [0, t_0]$ be the first time that γ reaches $\partial\Omega_2(z_0, 10/9\varepsilon)$.

$$
\begin{aligned}
L_{\mathrm{K}}^{\Omega}(\gamma) &= \int_0^{t_0} F_{\mathrm{K}}^{\Omega}(\gamma(t); \gamma'(t))\, dt \approx c \int_0^{t_0} \frac{|\gamma'_{N_P}(t)|}{\delta_\Omega(\gamma(t))}\, dt + c \int_0^{t_0} \frac{|\gamma'_{T_P}(t)|}{\sqrt{\delta_\Omega(\gamma(t))}}\, dt \\
&\geqslant c \int_0^{t_1} \frac{|\gamma'_{N_P}(t)|}{2\varepsilon}\, dt + c \int_0^{t_1} \frac{|\gamma'_{T_P}(t)|}{\sqrt{2\varepsilon}}\, dt \\
&\geqslant c \int_0^{t_1} \frac{|\gamma'_{T_P}(t)|}{\sqrt{2\varepsilon}}\, dt \\
&\geqslant \frac{c}{\sqrt{2\varepsilon}} \int_0^{t_1} |\gamma'_{T_P}(t)|\, dt.
\end{aligned}
$$

Let us define the curve

$$
\mu_1(t) = z_0 + \int_0^t \gamma'_{T_P}(s)\, ds, \qquad 0 \leqslant t \leqslant t_1,
$$

then

$$
\begin{aligned}
L_{\mathrm{K}}^{\Omega}(\gamma) &\geqslant \frac{c}{\sqrt{2\varepsilon}} \int_0^{t_1} |\mu_1'(t)|\, dt \\
&= \frac{c}{\sqrt{2\varepsilon}} L_{\mathrm{Euc}}(\mu_1) \geqslant \frac{c}{\sqrt{2\varepsilon}} \text{ length radius } Q_2(z_0, 10/9\varepsilon) \\
&= \frac{c}{\sqrt{2\varepsilon}} \sqrt{10/9\varepsilon} \geqslant \frac{c}{2},
\end{aligned}
$$

since the radius of $Q_2(z_0, 10/9\varepsilon)$ in the complex tangential direction is $\sqrt{10/9\varepsilon}$.

Case (ii). Let $t_2 \in [0, t_0]$ be the first time γ reaches $\partial Q_2(z_0, 10/9\varepsilon)$.

$$
\begin{aligned}
L_{\mathrm{K}}^{\Omega}(\gamma) &= \int_0^{t_0} F_{\mathrm{K}}^{\Omega}(\gamma(t); \gamma'(t))\, dt \approx c \int_0^{t_0} \frac{|\gamma'_{N_P}(t)|}{\delta_\Omega(\gamma(t))}\, dt + c \int_0^{t_0} \frac{|\gamma'_{T_P}(t)|}{\sqrt{\delta_\Omega(\gamma(t))}}\, dt \\
&\geqslant c \int_0^{t_2} \frac{|\gamma'_{N_P}(t)|}{2\varepsilon}\, dt \geqslant \frac{c}{2\varepsilon} \int_0^{t_2} |\gamma'_{N_P}(t)|\, dt.
\end{aligned}
$$

Let us define the curve

$$
\mu_2(t) = z_0 + \int_0^t \gamma'_{N_P}(s)\, ds, \qquad 0 \leqslant t \leqslant t_2,
$$

then

$$L_{\mathrm{K}}^{\Omega}(\gamma) \geq \frac{c}{2\varepsilon} \int_0^{t_2} |\mu_2'(t)|\, dt \geq \frac{c}{2\varepsilon} L_{\mathrm{Euc}}(\mu_2) \geq \frac{c}{2\varepsilon}\, 10/9\varepsilon \geq \frac{c}{2},$$

since the radius of $Q_2(z_0, 10/9\varepsilon)$ in the complex normal direction is $10/9\varepsilon$.

Let φ be a biholomorphic mapping from $Q_2(z_0, 10/9\varepsilon)$ onto $Q_0(0, 1) = \{z \in \Omega : |\pi_P(z)| < 1;\ |z| < 1\}$ such that $\varphi(z_0) = 0$ and $\varphi(w_0) = \mu_0$, and B^n is the unit ball in $\mathbb{C}^n$.

Then

$$K_{Q_2}(z_0, w_0) = K_{Q_0}(0; \mu_0) \geq K_{B^n}(0, u_0) = \frac{1}{2} \log \frac{1 + |u_0|}{1 - |u_0|} = M.$$

Since

$$K_{U \cap \Omega}(z_0, w_0) \leq K_{Q_2}(z_0, w_0)$$

there exists a constant $C = C(U, V, \Omega)$ such that

$$CK_{U \cap \Omega}(z_0, w_0) \leq L_{\mathrm{K}}^{\Omega}(\gamma),$$

where γ is any curve that escapes $Q_2(z_0, 10/9\varepsilon)$.

Hence

$$\begin{aligned} K_\Omega(z_0, w_0) &= \inf\{L_{\mathrm{K}}^{\Omega}(\beta) : \beta \text{ is a } C^1 \text{ curve connecting } z_0 \text{ with } w_0 \text{ in } \Omega\} \\ &= \min\{\inf_{\beta \text{ stays on } Q_2} L_{\mathrm{K}}^{\Omega}(\beta);\ \inf_{\beta \text{ escapes } Q_2} L_{\mathrm{K}}^{\Omega}(\beta)\} \\ &\geq \inf_{\beta \text{ escapes } Q_2} L_{\mathrm{K}}^{\Omega}(\beta) \geq CK_{U \cap \Omega}(z_0, w_0). \end{aligned}$$

REFERENCES

1. G. ALADRO, "Some Consequences of the Boundary Behavior of the Caratheodory and Kobayashi Metrics and Applications to Normal Holomorphic Functions," thesis, Pennsylvania State University, 1985.
2. S. G. KRANTZ, "Function Theory of Several Complex Variables," Wiley-Interscience, New York, 1982.
3. I. GRAHAM, Boundary behavior of the Caratheodory and Kobayashi metrics on strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary, *Trans. Amer. Math. Soc.* **207** (1975), 219–240.
4. R. I. ROYDEN, Remarks on the Kobayashi metric, *in* "Several Complex Variables, II, Proc. International Conf., University of Maryland, 1970," pp. 125–137, Lecture Notes in Math., Vol. 185, Spring-Verlag, Berlin, 1971.
5. N. SIBONY, A class of hyperbolic manifolds, *in* "Recent Developments in Several Complex Variables" (Fornaess, Ed.), pp. 357–372, Princeton Univ. Press, Princeton, 1981.